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Convex polyominoes and heaps of segments

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Abstract. We build a one-to-one correspondence between some polyominoes, called parallelogram polyominoes, and some heaps of segments. Then we derive explicit expressions for the generating function of convex polyominoes, according to their height, width and area. We also enumerate two subsets of convex polyominoes, namely the directed and convex polyominoes and the parallelogram polyominoes, according to these three parameters. We also study generating functions of polyominoes having a fixed width.

1. Introduction

Polyominoes are classical objects in combinatorics. They are also studied in physics as special cases of self-avoiding polygons that are used to model crystal growth or polymers. A polyomino is a finite union of elementary cells whose interior is connected (figure 1).

A polyomino P is column- (respectively row-) convex if the intersection of P with any vertical (respectively horizontal) line is connected. A convex polyomino is a polyomino which is both column- and row-convex (figure 2). We define three subclasses of convex polyominoes: parallelogram polyominoes (also called staircase polyominoes, figure 3), stack polyominoes (also called pyramid polyominoes, figure 4), and directed and convex polyominoes (figure 5). One or two corners of such polyominoes are also corners of their bounding rectangle.

The perimeter, or height and width generating functions of these subsets of polyominoes are algebraic functions (see Temperley 1952, Polya 1969, Delest and Viennot 1984, Lin and Chang 1988). Some area generating functions were given for stack polyominoes (Temperley 1952), and for parallelogram polyominoes (Delest and Fedou 1989, Brak and Guttmann 1990, Lin and Tzeng 1991). For the area generating function of convex polyominoes, only asymptotic results had been found (Klarner and Rivest 1974, Bender 1974), until Lin (1991) gave a first exact formula.

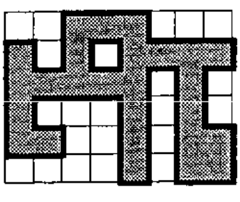

Figure 1. A polyomino.

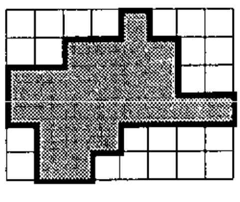

Figure 2. A convex polyomino.

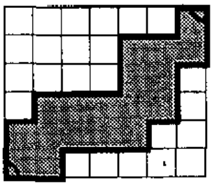

Figure 3. A parallelogram polyomino.

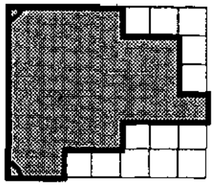

Figure 4. A stack polyomino.

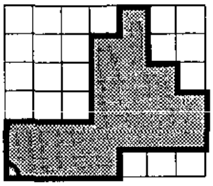

Figure 5. A directed and convex polyomino.

We give here the generating function of parallelogram polyominoes, directed and convex polyominoes, together with a new formula for the generating function of convex polyominoes, according to their height, width and area. Our work is based on a one-to-one correspondence between parallelogram polyominoes and some heaps of segments; it uses various decompositions of convex polyominoes into parallelogram polyominoes and stack polyominoes (see Bousquet-Mélou and Viennot 1990, Bousquet-Mélou 1991). Note that a second formula for the generating function of convex polyominoes can be obtained in a different way (Bousquet-Mélou 1990, 1991, 1992).

Constructing a one-to-one correspondence between the objects one wishes to enumerate and some heaps of pieces (these pieces can be segments, but also many geometric or algebraic structures) is of special interest: the set of heaps of pieces has algebraic properties that often allows us to write theorems, called inversion theorems, that give the generating functions of some subsets of heaps.

This method appears to be successful in the case of convex polyominoes, but was also used previously by Viennot to enumerate directed animals on the square lattice (Viennot 1985, Gouyou-Beauchamps and Viennot 1988). Directed animals have been studied by many physicists. Some other exact results about their enumeration were given by Derrida *et al* (1982), Hakim and Nadal (1983) and Dhar (1982, 1983).

Notation. Let the generating function of a given subset $\mathscr{S}$ of polyominoes be

$$P(x, y, q) = \sum_{n,m,a} x^n y^m q^a P_{n,m,a}$$

where $P_{n,m,a}$ is the number of polyominoes of $\mathscr{S}$ having width n, height m and area a. We use the standard notations: if $n \geqslant 1$,

$$(a)_n = (1-a)(1-aq)\dots(1-aq^{n-1}).$$

Let $[n]!$ denote the classical q-analogue of $n!$, that is $(q)_n/(1-q)^n$. By convention, $(a)_n = 1$ and $[n]! = 1$ if $n \leqslant 0$.

2. Parallelogram polyominoes and heaps of segments

The theory of heaps of pieces was developed by Viennot (1986) and is equivalent to Cartier and Foata's partially commutative monoids theory (1969). Heaps of segments are an example of this theory.

Intuitively, a heap of segments is built by putting a finite number of solid segments one upon the other. Each segment has a finite level $n \geq 0$ and a position $[a, b]$, with $1 \leq a \leq b$, and lies either on the horizontal axis or upon some part of another segment having level $n-1$ (figure 6). Let E and F be two heaps of segments. The superposition of E on F is obtained by putting E above F, and is denoted $F \bowtie E$.

A segment of a heap is minimal when it has zero level, and maximal when no other segment covers it—even partially. A heap is said to be trivial when all its pieces are minimal (and maximal).

A semi-pyramid is a heap of segments having a unique maximal segment, whose position is $[1, b]$, where $b \geq 1$.

We put on a segment S of position $[a, b]$ the weight $\nu(S) = xy^{b-a}q^b$, and on a heap E composed of the segments $S_1, \ldots, S_n$ the weight $\nu(E) = \Pi_{1 \leq i \leq n}\, \nu(S_i)$.

Then, we describe a one-to-one correspondence f between the parallelogram polyominoes and the semi-pyramids such that, if P is a parallelogram polyomino having width n, height m and area a, $\nu(f(P)) = x^n y^{m-1} q^a$.

Briefly, let P be a parallelogram polyomino, and $C_1, \ldots, C_n$ be its columns (from left to right). For $1 \leq i \leq n$, let b_i be the number of cells of C_i. Let $a_1 = 1$, and, for $2 \leq i \leq n$, let a_i be the number of cells by which the columns C_{i-1} and C_i are 'glued'. Then, we build $f(P)$ by placing a segment having position $[a_n, b_n]$, then a segment having position $[a_{n-1}, b_{n-1}]$, and so on until we place the last segment, whose position is $[a_1, b_1]$ (figure 7).

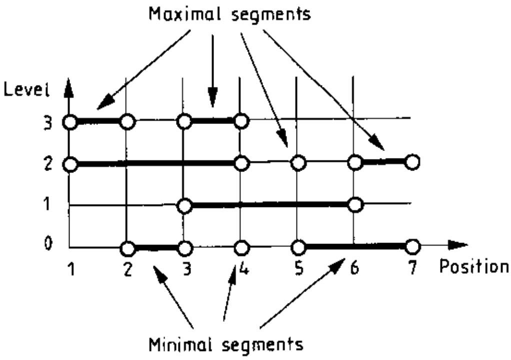

Figure 6. A heap of segments.

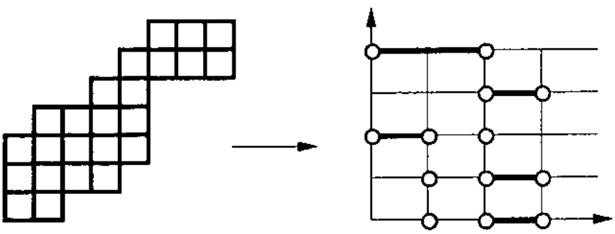

Figure 7. A parallelogram polyomino and the associated semi-pyramid.

Such a bijection is of special interest because there exist some inversion theorems that give the generating functions of the heaps of segments E satisfying one or both the two following conditions: (i) every maximal segment of E has its position in a given set $\mathcal{M}$; (ii) every minimal segment of E has its position in a given set $\bar{\mathcal{M}}$.

For example, the generating function of heaps of segments satisfying (i) is

$$\sum_{E/\max(E)\subset\mathcal{M}} \nu(E) = \frac{\sum_{F\in\mathcal{T}({}^c\mathcal{M})} (-1)^{|F|}\nu(F)}{\sum_{F\in\mathcal{T}} (-1)^{|F|}\nu(F)} \tag{1}$$

where $\mathcal{T}$ is the set of trivial heaps and $\mathcal{T}({}^c\mathcal{M})$ is the set of trivial heaps having no segment in $\mathcal{M}$. We denote by $|F|$ the number of segments of the heap F.

Let us prove this result. We have

$$\left(\sum_{E/\max(E)\in\mathcal{M}} \nu(E)\right)\left(\sum_{F\in\mathcal{T}} (-1)^{|F|}\nu(F)\right) = \sum_{E'} \left(\nu(E') \sum_{\substack{(E,F)\\ E'=F\boxtimes E}} (-1)^{|F|}\right) \tag{2}$$

where the first sum is taken over all the heaps E' that can be obtáined by putting a heap E having all its maximal segments in $\mathcal{M}$ upon a trivial heap F, and the second one is taken over all such heaps E and F satisfying $E' = F\boxtimes E$. The heap E' being given, we notice that F must be included in the set of minimal segments of E', and must contain any minimal segment of E' that is simultaneously maximal and not in $\mathcal{M}$. Then, the choice of F determines uniquely the heap E. But

$$\sum_{\min(E')\cap\max(E')\cap{}^c\mathcal{M}\subset F\subset\min(E')} (-1)^{|F|} = 0$$

unless every minimal segment of E' is also maximal and not in $\mathcal{M}$, which means that E' is trivial and has no segment in $\mathcal{M}$. In this case, $F = E'$ and E is the empty heap.

Then, we prove that the 'alternating' generating function of trivial heaps is:

$$\sum_{F\in\mathcal{T}} (-1)^{|F|}\nu(F) = \sum_{n\geqslant 0} \frac{(-1)^n x^n q^{\binom{n+1}{2}}}{(q)_n (yq)_n}. \tag{3}$$

We can now derive the generating function $X(x, y, q)$ of parallelogram polyominoes from theorem (1), since a semi-pyramid is a heap having all its maximal segments in the set $\mathcal{M} = \{[1, n], n \geqslant 1\}$. We get:

$$X = -y\frac{\hat{N}(x)}{N(x)} \tag{4}$$

where

$$N(x) = \sum_{n\geqslant 0} \frac{(-1)^n x^n q^{\binom{n+1}{2}}}{(q)_n (yq)_n} \tag{5}$$

and

$$\hat{N}(x) = \sum_{n\geqslant 1} \frac{(-1)^n x^n q^{\binom{n+1}{2}}}{(q)_{n-1} (yq)_n}. \tag{6}$$

Remarks. (1) The width and area generating function of parallelogram polyominoes (special case $y = 1$) has been given by Delest and Fedou (1989). The perimeter and area generating function (case $x = y$) was found by Brak and Guttmann (1990). Recently, Lin and Tzeng (1991) generalized their result and obtained the width, height and area generating function of parallelogram polyominoes. Their formula looks at first sight more complicated than ours, but it is quite easy to check it is, in fact, the same one.

(2) In the case $y=1$, the series X is the quotient of two q-analogues of Bessel functions:

$$X(x,1,q)=(1-q)\frac{I_1(xq/(1-q)^2;q)}{I_0(xq/(1-q)^2;q)} \tag{7}$$

where

$$I_m(t;q)=\sum_{n\geq 0}\frac{(-1)^n t^{n+m} q^{\binom{n+m}{2}}}{[n]![n+m]!}. \tag{8}$$

3. Directed and convex polyominoes

We show that a directed and convex polyomino can be split into a parallelogram polyomino P_1 and a stack polyomino P_2, so that if P_2 has height n, then the first column of P_1 has height $n+1$ (figure 8).

Thus, to enumerate directed and convex polyominoes is equivalent to enumerating:

- first, parallelogram polyominoes whose first column has a given height;
- and then, stack polyominoes with a given height.

We use again theorem (1) to solve the first problem: the first column of a parallelogram polyomino has height n if and only if the maximal segment of the associated semi-pyramid has position $[1,n]$. Thus, the generating function of parallelogram polyominoes having their first column of height n is

$$xy^n q^n \frac{N(xq^n)}{N(x)} \tag{9}$$

where $N(x)$ is the series defined by (5).

The following classical expression of the generating function $T(x,y,q)$ of stack polyominoes is linked to the enumeration of Ferrers diagrams (see Andrews 1976):

$$T=\sum_{m\geq 1}\frac{x^m y q^m}{(yq)_{m-1}(yq)_m}. \tag{10}$$

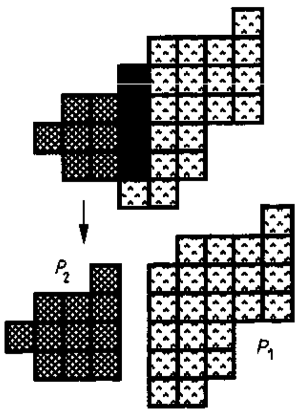

Figure 8. Decomposition of a directed and convex polyomino.

Then, we prove easily that the generating function of stack polyominoes of height n is given by

$$\frac{xy^n q^n T_n}{(xq)_n} \tag{11}$$

where T_n denotes a polynomial in the two variables x and q, with positive integer coefficients, defined by the following recurrence relations:

$$\begin{aligned} &T_0 = 1 \qquad T_1 = 1 \\ &T_n = 2T_{n-1} + (xq^{n-1} - 1)T_{n-2} \qquad \text{if } n \geqslant 2. \end{aligned} \tag{12}$$

Combining (9) and (11) (and also using (10) in the computation), we finally obtain the generating function $Y(x, y, q)$ of directed and convex polyominoes:

$$Y = y\frac{R(x) - \hat{N}(x)}{N(x)} \tag{13}$$

where $N(x)$ and $\hat{N}(x)$ are the series defined by (5) and (6) respectively and

$$R(x) = y \sum_{n \geqslant 2} \left[\frac{x^n q^n}{(yq)_n} \left(\sum_{m=0}^{n-2} \frac{(-1)^m q^{\binom{m+2}{2}}}{(q)_m (yq^{m+1})_{n-m-1}} \right) \right]. \tag{14}$$

4. Convex polyominoes

Let P be a convex polyomino and R be the smallest rectangle containing P. Let [N, N′] (respectively [W, W′], [S, S′], [E, E′]) be the intersection of P with the upper (respectively left, lower, right) border of R, the points N, N′, W, W′, S, S′, E, E′ being taken counterclockwise (figure 9).

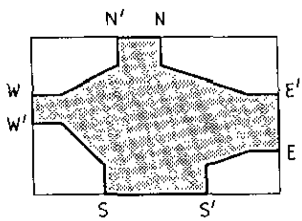

Figure 9. A convex polyomino.

We define three subsets of convex polyominoes. Let $\mathscr{A}$ be the set of convex polyominoes such that the vertical line passing by N is at the right of the vertical line passing by S. Let $\mathscr{A}'$ be the set of convex polyominoes such that the vertical line passing by S′ is at the right of the vertical line passing by N′ (figure 10). Let $\mathscr{B}$ be the intersection of $\mathscr{A}$ and $\mathscr{A}'$.

Note that the symmetric, up to any vertical axis, of a polyomino of $\mathscr{A}$ is a polyomino of $\mathscr{A}'$ (and vice-versa), and that the union of $\mathscr{A}$ and $\mathscr{A}'$ is the set of convex polyominoes.

These remarks imply that the generating function $Z(x, y, q)$ of convex polyominoes is

$$Z = 2A - B \tag{15}$$

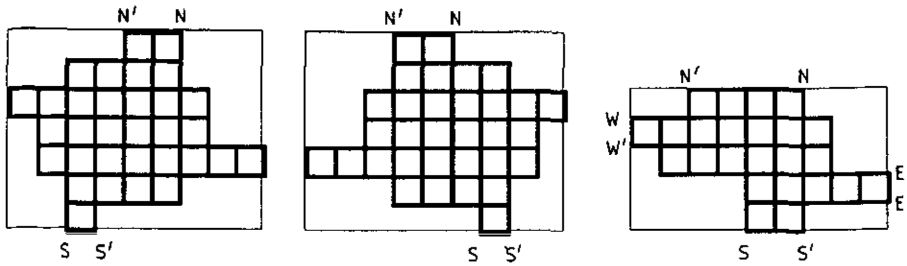

Figure 10. Elements of (left to right) $\mathscr{A}$, $\mathscr{A}'$ and $\mathscr{B}$.

where $A(x, y, q)$ (respectively $B(x, y, q)$) is the generating function of the convex polyominoes of $\mathscr{A}$ (respectively $\mathscr{B}$).

We derive from (11) that the generating function of the polyominoes of $\mathscr{B}$ is

$$B = \sum_{n \geq 1} \frac{xy^n q^n (T_n)^2}{(xq)_{n-1}(xq)_n} \tag{16}$$

where T_n is the polynomial defined by (12).

Let P be a polyomino of $\mathscr{A}$. It can be divided (figure 11) into three polyominoes, a parallelogram polyomino P_1 and two stack polyominoes P_2 and P_3. If P_2 (respectively P_3) has height n (m), then the first (last) column of P_1 has height $n+1$ ($m+1$).

Thus, to enumerate convex polyominoes is equivalent to enumerating:

- on the one hand, parallelogram polyominoes whose first and last columns have given heights,
- on the other hand, stack polyominoes of given height.

We already solved this second problem when we enumerated directed and convex polyominoes (see (11)). Once more, we use our bijection between parallelogram polyominoes and semi-pyramids and a new inversion theorem to solve the first problem: the generating function of parallelogram polyominoes whose first (last) column has height n (m) is

$$x^2 y^n q^{m+n} N_m \frac{N(xq^n)}{N(x)} - yN_m^n \tag{17}$$

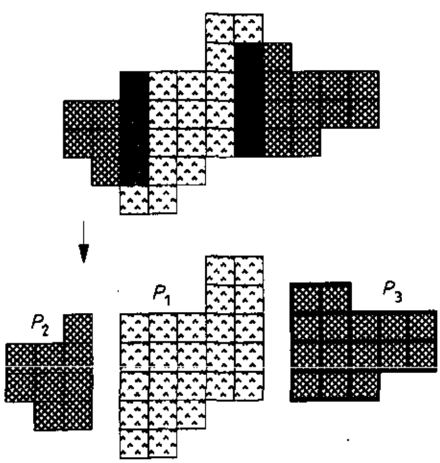

Figure 11. Decomposition of a polyomino of $\mathscr{A}$.

where $N(x)$ is the series given by (5) and N_m and N_m^n are polynomials in the three variables x, y and q, defined as follows:

$$\begin{aligned} &N_0 = 0 \qquad N_1 = 1 \\ &N_n = (1 + y - xq^{n-1})N_{n-1} - yN_{n-2} \qquad \text{if } n \geqslant 2 \end{aligned} \tag{18}$$

$$\begin{aligned} &N_m^n = 0 \qquad \text{if } m < n \\ &N_n^n = -xy^{n-1}q^n \\ &N_m^n = x^2y^{n-1}q^{n+m}N_{m-n}(xq^n) \qquad \text{if } n < m. \end{aligned} \tag{19}$$

Combining (11), (16) and (17), we finally obtain the generating function $Z(x, y, q)$ of convex polyominoes:

$$Z = 2y\frac{(R(x) - \hat{N}(x))V(x)}{N(x)} - 2yM(x) - B(x) \tag{20}$$

where

$$V = \sum_{m \geqslant 0} xq^{m+1}\frac{T_m N_{m+1}}{(xq)_m} \tag{21}$$

$$M = \sum_{0 \leqslant n \leqslant m} N_{m+1}^{n+1}\frac{T_m T_n}{(xq)_m (xq)_n} \tag{22}$$

the polynomials T_n, N_m and N_m^n are defined by (12), (18) and (19) and the series $N(x)$, $\hat{N}(x)$, $R(x)$ and $B(x)$ by (5), (6), (14) and (16) respectively.

5. Convex polyominoes having a fixed width

We study more thoroughly generating functions of polyominoes having a fixed width. These functions are rational series in y and q. We use a correspondence between some heaps of segments and some binary trees to get the following results, where $\lfloor n/i \rfloor$ denotes the integer part of n/i.

The generating function of parallelogram polyominoes having width n is

$$x^n yq^n \frac{\bar{X}_n(y, q)}{\prod_{k=1}^n (1 - yq^k)^{\lfloor \frac{n}{k} \rfloor}} \tag{23}$$

where $\bar{X}_n$ is a polynomial in the variables y and q with integer coefficients. Moreover,

$$\bar{X}_1(y, 1) = 1 \tag{24}$$

and, if $n \geqslant 2$,

$$\bar{X}_n(y, 1) = (1 - y)^{\sum_{k=2}^n (\lfloor \frac{n}{k} \rfloor - 1)} \sum_{k=0}^{n-2} y^k \frac{1}{n-1}\binom{n-1}{k}\binom{n-1}{k+1}. \tag{25}$$

The generating function of directed and convex polyominoes having width n is

$$x^n yq^n \frac{\bar{Y}_n(y, q)}{\prod_{k=1}^n (1 - yq^k)^{\lfloor \frac{n-1}{k} \rfloor + 1}} \tag{26}$$

where $\bar{Y}_n$ is a polynomial in the variables y and q with integer coefficients. Moreover,

$$\bar{Y}_n(y, 1) = (1 - y)^{\sum_{k=2}^{n-1} \lfloor \frac{n-1}{k} \rfloor} \sum_{k=0}^{n-1} y^k \binom{n-1}{k}^2. \tag{27}$$

The generating function of convex polyominoes having width n is

$$x^n y q^n \frac{\bar{Z}_n(y, q)}{\prod_{k=1}^{n-1} (1-yq^k)^{\lfloor \frac{n-2}{k} \rfloor + 2}(1-yq^n)} \tag{28}$$

where $\bar{Z}_n$ is a polynomial in the variables y and q with integer coefficients. Moreover,

$$\bar{Z}_n(y, 1) = (1-y)^{\sum_{k=1}^{n-2} \lfloor \frac{n-2}{k} \rfloor} \sum_{k=0}^{n} y^k \left[\frac{n(1+k)-2k}{n} \binom{2n}{2k} - 2n \binom{n-1}{k-1} \binom{n-1}{k} \right]. \tag{29}$$

When enumerating those polyominoes according only to their area and width (special case $y=1$), we get simpler results by studying directly the formulae (4), (13) and (20).

The width and area generating function of parallelogram polyominoes having width n is

$$x^n q^n \frac{\hat{X}_n}{(q)_{n-1}(q)_n} \tag{30}$$

where $\hat{X}_n$ is a polynomial in the variable q with integer coefficients.

The width and area generating function of directed and convex polyominoes having width n is

$$x^n q^n \frac{\hat{Y}_n}{(q)_{n-1}(q)_n} \tag{31}$$

where $\hat{Y}_n$ is a polynomial in the variable q with integer coefficients.

The width and area generating function of convex polyominoes having width n is

$$x^n q^n \frac{\hat{Z}_n}{[n-2]!(q)_{n-1}(q)_n} \tag{32}$$

where $\hat{Z}_n$ is a polynomial in the variable q with integer coefficients.

We do not have any explicit formula for the numerators $\hat{X}_n$, $\hat{Y}_n$ and $\hat{Z}_n$, but we were able to evaluate the first values, thanks to the formal calculus system MAPLE. Those results are remarkable: we conjecture that these polynomials have positive coefficients and are unimodal. The $\hat{X}_n$ polynomials seem to be symmetric.

Examples. The first values of the $\hat{X}_n$ polynomials are:

$$\hat{X}_1 = 1 \qquad \hat{X}_2 = 1 \qquad \hat{X}_3 = 1 + q + q^2 + q^3$$

$$\hat{X}_4 = 1 + 2q + 4q^2 + 6q^3 + 7q^4 + 6q^5 + 4q^6 + 2q^7 + q^8$$

$$\hat{X}_5 = 1 + 3q + 8q^2 + 17q^3 + 30q^4 + 45q^5 + 58q^6 + 66q^7 + 66q^8 + 58q^9 + 45q^{10} + 30q^{11} + 17q^{12} + 8q^{13} + 3q^{14} + q^{15}.$$

The first values of the $\hat{Y}_n$ polynomials are:

$$\hat{Y}_1 = 1 \qquad \hat{Y}_2 = 1 + q \qquad \hat{Y}_3 = 1 + 3q + 3q^2 + 2q^3 + q^4$$

$$\hat{Y}_4 = 1 + 5q + 9q^2 + 14q^3 + 18q^4 + 17q^5 + 13q^6 + 7q^7 + 3q^8 + q^9$$

$$\hat{Y}_5 = 1 + 7q + 17q^2 + 37q^3 + 70q^4 + 109q^5 + 147q^6 + 173q^7 + 180q^8 + 165q^9 + 133q^{10} + 94q^{11} + 57q^{12} + 29q^{13} + 12q^{14} + 4q^{15} + q^{16}.$$

The first values of the $\hat{Z}_n$ polynomials are:

$$\hat{Z}_1=1 \qquad \hat{Z}_2=1+2q+q^2 \qquad \hat{Z}_3=1+6q+12q^2+12q^3+7q^4+2q^5$$

$$\hat{Z}_4=1+11q+43q^2+95q^3+150q^4+186q^5+181q^6+137q^7+79q^8+33q^9+10q^{10}+2q^{11}.$$

Fedou (1991) has just shown that the $\hat{X}_n$ polynomial is the generating function of certain braids. As a corollary, he shows that $\hat{X}_n$ has positive coefficients and is symmetric.

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